U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE

MARCH 1971

NUMERICAL INSTABILITY OF A MIXED IMPLICIT-EXPLICIT INTEGRATION PROCEDURE

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OFFICE NOTE 55

1. Introduction

In a recent Office Note (#53), an account was given of our investigations into the stability characteristics of the 'modified' semi-implicit method. The basic concept of this method is the separation of the external mode (treated implicitly) from the internal modes (treated explicitly). The failure to satisfactorily achieve this separation has been reported in Office Note 53, which serves as the obituary for the approach.

There were some loose ends, however: we stated that the apparent instability was associated with the internal computational mode, which for reasons unknown amplifies very rapidly for increasing Δt. Moreover, in the ζ-plane representation, the root which corresponds to the internal computational mode remains on the real axis for all Δt.[1]

Shuman has suggested that this behavior might be caused by not clearly separating the external and internal modes. In particular, he argued that the treatment outlined in Section 4 of #53 might result in the internal mode being treated explicitly with respect to the thermodynamic equation, but implicitly in the momentum equations. We perhaps could have seen this, had we been able to solve analytically the eighth-order polynomial in ζ, the time dependent part of the solution, resulting from the requirement that the determinant of the coefficient matrix vanish. Unfortunately, this does not seem tractable.

As a means of avoiding this difficult problem, Shuman has pointed out that one might examine the stability properties of a mixed implicit-explicit integration scheme applied to a system which permits only an external mode. The purpose of this note is to indicate the results of carrying out this suggestion.

2. Linear Analysis of a Mixed Implicit-Explicit System

Consider the linear equations

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0\ , \qquad (2)$$

which govern the behavior of a homogeneous, incompressible fluid under the

[1] This means that the amplification occurs at zero phase; i.e., the amplifying disturbance is stationary, but not the mean as reported in Office Note 53.

restoring influence of gravity. Now, if the supposition that the instability reported in Office Note 53 is due to a mixed implicit-explicit approximation to one mode is, indeed, valid, then it would be anticipated that a similar approximation of Eqns. (1) and (2) would yield a similar instability.

The time derivatives will be approximated by the 'leap-frog' scheme, thus admitting a computational mode. Spatial derivatives will be treated analytically for the purposes of this analysis. In order to simulate the mixed implicit-explicit approximation, we choose to approximate the height-gradient term in Eqn. (1), and the divergence term in Eqn. (2), *explicitly*:

$$u^{n+1} - u^{n-1} + g\Delta t(h^{n+1} + h^{n-1})_x = 0 \tag{3}$$

$$h^{n+1} - h^{n-1} + 2H\Delta t(u^n)_x = 0 \tag{4}$$

where spatial differentiation is indicated by a subscript "x". We now assume solutions of the form

$$q = q_o \zeta^n e^{ikx} \tag{5}$$

Stability requires $|\zeta| \le 1$; we now enquire as to whether solutions exist for this criterion, and under what circumstances.

Substitution of Eqn. (5) into Eqns. (3) and (4) yields the system

$$(\zeta^2-1)u + \{gik\Delta t(\zeta^2+1)\}h = 0 \tag{6}$$

$$(\zeta^2-1)h + \{2ik\Delta tH\zeta\}u = 0 \tag{7}$$

The determinant of this system must vanish, giving the frequency equation

$$(\zeta^2-1)^2 + 2(k\Delta t)^2 \, gH\zeta(\zeta^2+1) = 0 \tag{8}$$

or, defining $\varepsilon = k\Delta t$, $c^2 = gH$, this may be rewritten as

$$\zeta^4 + 2\varepsilon^2c^2\zeta^3 - 2\zeta^2 + 2\varepsilon^2c^2\zeta + 1 = 0 \tag{9}$$

Now, if we expand the product $(\zeta^2+a\zeta+1)(\zeta^2+b\zeta+1) = 0$, we find

$$\zeta^4 + (a+b)\zeta^3 + (2+ab)\zeta^2 + (a+b)\zeta + 1 = 0 \tag{10}$$

Comparison of the coefficients of each term of Eqns. (9) and (10) reveals the following values of a, b:

$$a = - \frac{8}{(2\varepsilon^2c^2 \pm \sqrt{(2\varepsilon^2c^2)^2+16}} = - \frac{8}{D} \tag{11}$$

$$b = \frac{2\varepsilon^2c^2 \pm \sqrt{(2\varepsilon^2c^2)^2 + 16}}{2} = \tfrac{1}{2} D \tag{12}$$

where $D = 2\varepsilon^2c^2 \pm \sqrt{(2\varepsilon^2c^2)^2 + 16}$. Eqn. (9) may be expressed as

$$\{ \zeta^2 - \left(\frac{8}{D}\right)\zeta + 1\}\{ \zeta^2 + \tfrac{1}{2} D + 1\} = 0 \tag{13}$$

The two quadratic factors of Eqn. (13) yield the following expressions for ζ:

$$\zeta = \tfrac{1}{2} \{\frac{8}{D} \pm \sqrt{\frac{64}{D^2} - 4}\ \} \tag{14}$$

$$\zeta = \tfrac{1}{4} \{- D \pm \sqrt{D^2 - 16}\ \} \tag{15}$$

With respect to Eqn. (14), if

$$\frac{64}{D^2} < 4 \tag{16}$$

then the radical is negative, and (14) may be rewritten as

$$\zeta = \tfrac{1}{2}\{ \frac{8}{D} \pm i \sqrt{4 - \frac{64}{D^2}}\ \} \tag{17}$$

and the magnitude of ζ is

$$|\zeta| = \{ \tfrac{1}{4} [\frac{64}{D^2} + 4 - \frac{64}{D^2}]\}^{\frac{1}{2}} = 1 \tag{18}$$

so that computational neutrality is assured if the criterion (16) is satisfied. If not, ζ is a pure real number, and its magnitude is

$$|\zeta| = \{ \tfrac{1}{4}[\frac{64}{D^2} + \frac{64}{D^2} - 4] \}^{\frac{1}{2}} > 1 \tag{19}$$

which clearly demonstrates instability.

Similar arguments may be advanced with respect to Eqn. (16): if the inequality

$$D^2 < 16 \tag{20}$$

is satisfied, ζ is complex with magnitude unity; if not, ζ is real with magnitude greater than unity.

Now, the inequality (16) may be rewritten as

$$D^2 > 16 \tag{21}$$

and it is immediately obvious that both (20) and (21) cannot simultaneously be satisfied. Therefore, *the solutions corresponding to either (14) or (15) will be absolutely unstable*, irrespective of the value of Δt. Moreover, the unstable solutions will always lie on the real axis of the complex ζ-plane.

This result strongly supports Shuman's contention that the implicit method outlined in Section 4 of Office Note 53 does not achieve completely explicit treatment of the internal gravity mode. It thus appears that we may have buried a live idea prematurely.